Friedmann equation:

In the classical approach:

Due to the homogeneity of the universe, we can choose a reference system OXYZ. We shall further take a mass δm at a distance $\vec{r}$ from the origin of the reference system, that moves together with the space around it (the shell or radius r), with speed $\vec{v} = H\vec{r}$. From Gauss Theorem, only the mass inside the shell generates gravitational field on the border (mass δm).

$\oiint_{\partial V} \vec{g}\, d\vec{A} = -4\pi G M_V$

From Newton's second principle we have :

$\frac{G\delta m\, M_V}{r^2} = -\delta m\, \ddot{r}$ (8)

$\frac{G\delta m\, M_V}{r^2}\, dr = -\delta m\, \ddot{r}\, dr = -\delta m\, \ddot{r}\, \dot{r}\, dt = -\frac{1}{2}\delta m\, d(\dot{r}^2)$

After integration we obtain

$\frac{\dot{r}^2}{2} = \frac{G M_V}{r} + C$

We know that $\dot{r} = Hr$ and $M = \frac{4\pi}{3} r^3 \rho_m$, where ρ_m is the mass density

$\frac{H^2 r^2}{2} = \frac{4\pi G \rho_m}{3r} r^3 + C$

$\left(\frac{H^2}{2} - \frac{4\pi G \rho_m}{3}\right) r^2 = C$ (9)

Let $K \equiv -\frac{2C}{c^2}$, where $c =$ speed of light in vacuum

If we derived the Friedmann equation using general relativity the numerical constant K would have told us the curvature of the space .

$$\text{If} : k = \begin{cases} 1, \text{the geometry would be spherical} \\ 0, \text{the geometry would be flat (euclidian)} \\ -1, \text{the geometry would be hyperbolic} \end{cases}$$

$\left(H^2 - \frac{8\pi G \rho_m}{3}\right) r^2 = -K c^2$

$\vec{r} = a\vec{r}_0 \Rightarrow \left(H^2 - \frac{8\pi G \rho_m}{3}\right) a^2 = \frac{-\mathrm{Kc}^2}{r_0^2}$ (10)

This is the Friedmann equation.

Density as a function of the scale factor:

Since the universe is homogeneous and isotropic, the temperature is constant throughout the space at any given time.

From the first principle of thermodynamics, we have $dU = \delta Q - \delta L$, but from above, $\delta Q = 0$, so $dU = -p dV$

$dV = \frac{4\pi}{3} d(r^3)$

But $U = \frac{4\pi}{3} \rho_e r^3$, where ρ_e is the mass energy density

Thus, $d\left(\frac{4\pi}{3} \rho_e r^3\right) = -p \frac{4\pi}{3} d(r^3)$

$\frac{d(\rho_e r^3)}{dt} = \frac{-p\, d(r^3)}{dt}$ (11) , which is the fluid equation

The equation of state : $p = w\rho_e$ (12)

Substituting p with $w\rho_e$ in (11):

$\frac{d(\rho_e r^3)}{dt} + \frac{w\rho_e\, d(r^3)}{dt} = 0$

$\frac{d\rho_e}{dt} r^3 + \rho_e \frac{d(r^3)}{dt} + \frac{w\rho_e\, d(r^3)}{dt} = 0$

$\frac{d\rho_e}{dt} r^3 + \frac{(1+w)\rho_e\, d(r^3)}{dt} = 0$

$\frac{d\rho_e}{\rho_e} + (1+w)\frac{d(r^3)}{r^3} = 0$ (13)

After integration:

$\ln\left(\rho_e\, r^{3(1+w)}\right) = \mathrm{ct} \Rightarrow \rho_e r^{3(1+w)} = \rho_{0,e} r_0^{3(1+w)}$

$r = a r_0$ and $a_0 = 1$, so $\rho_e = \rho_{0,e} a^{-3(1+w)}$ (14)

Due to $\rho_e = \rho_m c^2$, it also holds in the form $\rho_m = \rho_{0,m} a^{-3(1+w)}$ (14')

Thus, $\rho_e \sim a^{-3(1+w)}$

Equation of state for matter and radiation:

Let there be a box of length L and area S.

Let there be a generic particle with impulse p and p_x, p_y, p_z the projections of the impulse on axes OX, OY, OZ.

$\langle p_x^2 \rangle = \langle p_y^2 \rangle = \langle p_z^2 \rangle$ (15)

$p_x^2 + p_y^2 + p_z^2 = p^2 \Rightarrow \langle p_x^2 + p_y^2 + p_z^2 \rangle = \langle p^2 \rangle \Rightarrow \langle p_x^2 \rangle + \langle p_y^2 \rangle + \langle p_z^2 \rangle = \langle p^2 \rangle \Rightarrow \langle p_x^2 \rangle = \langle p_y^2 \rangle = \langle p_z^2 \rangle = \frac{1}{3}\langle p^2 \rangle$ (15')

The variation of impulse at a collision is $\delta p_x = 2p_x$

The pressure $P = \frac{F}{S} = \frac{dp_{\text{collision}}}{S\,\mathrm{dt}} = \frac{1}{S}\left\langle \frac{\delta N}{\delta t} 2p_x \right\rangle = \frac{1}{S}\langle \frac{n L S v_x}{2L} 2p_x \rangle = n\langle p_x v_x \rangle = n\langle p_x v_x \frac{mc^2}{E} \rangle = n\langle \frac{p_x m v_x c^2}{E} \rangle = n\langle \frac{p_x^2 c^2}{E} \rangle = n\,\frac{c^2}{E}\frac{\langle p^2 \rangle}{3}$ (16)

For normal matter:

$P = n\frac{m^2 c^2}{E\left(\frac{1}{v^2} - \frac{1}{c^2}\right)} \Rightarrow \frac{P}{\rho} = \frac{n p^2 c^2}{3nE^2} = \frac{1}{3}\frac{m^2 v^2 c^2}{m^2 c^4} = \frac{1}{3}\frac{v^2}{c^2} \simeq 0$, (dust approximation) thus $w = 0$

For photons:

$P = n\frac{p^2 c^2}{3E} = \frac{nE^2}{3E} = \frac{nE}{3} = \frac{\rho_e}{3}$, thus $w = \frac{1}{3}$

(concluzii personale)

We can observe from (16) that for particles that $P = n\frac{c^2}{E}\frac{\langle p^2 \rangle}{3} = \frac{nE}{E}\frac{c^2}{E}\frac{\langle p^2 \rangle}{3} = \rho_e \frac{m^2 c^2 v^2}{3E^2} = \frac{\rho_e}{3}\frac{v^2}{c^2}$, so $w = \frac{P}{\rho_e} = \frac{1}{3}\frac{v^2}{c^2}$, thus it can only be between 0 and $\frac{1}{3}$.

If there would be a particle fluid with $w > \frac{1}{3}$, its particles' speed would be greater than the speed of light.

Thus, a fluid with $w > \frac{1}{3}$ can only be intrinsic to the space.

Fluids where $w < 0$ must have negative pressure.

If such fluids would be made of particles, they would receive impulse in the direction of their original velocity.

It is obvious that if such particles would have positive mass, their impulse would grow indefinitely. As such, the only possibility would be for them to have negative mass (considering that the energy density ρ_e must be positive for w to be negative, their energy must be positive, so it would be defined as $E = |mc^2|$), which would explain models of universes with $0 > w > -\frac{1}{3}$.

For fluids with $w < -\frac{1}{3}$ it is clear that they cannot be made of particles, thus they must be intrinsic to the space.

By writing the first principle of thermodynamics in this case we obtain:

$dL = -p dV = -w\rho_e dV = |w|\rho_e dV \Rightarrow \frac{dL}{dV} = |w|\rho_e = |w|\frac{dE}{dV}$

For $w > -1$ the fluid receives less energy than it spends expanding, so its density lowers with the increase of the universe's size.

For $w < -1$ the fluid receives more energy than it spends expanding, so its density grows with the increase of the universe's size.

For $w = -1$ the fluid receives exactly the same energy it spends expanding, thus having constant density no matter the size of the universe or how it grows or shrinks. Such a result can be interpreted as the energy of the empty space.

Expansion of single-fluid universe:

From (10):

$\left(H^2 - \frac{8\pi G\rho_m}{3}\right)a^2 = \frac{-Kc^2}{r_0^2}$

$\left(H^2 - \frac{8\pi G\rho_m}{3}\right)a^3 = \frac{-Kc^2}{r_0^2}a$

$\left(H^2 - \frac{8\pi G\rho_m}{3}\right)a^3 = \frac{-Kc^2}{r_0^2}a$ (17)

$\frac{d}{dt}\left[\left(H^2 - \frac{8\pi G\rho_m}{3}\right)a^3\right] = \frac{d}{\mathrm{dt}}\left(\frac{-Kc^2}{r_0^2}a\right)$, substituting H with $\frac{\dot{a}}{a}$, we obtain

$\frac{d}{\mathrm{dt}}\left(\dot{a}^2 a - \frac{8\pi G\rho_m a^3}{3}\right) = \frac{d}{\mathrm{dt}}\left(\frac{-Kc^2}{r_0^2}a\right)$

$2a\,\dot{a}\,\ddot{a} + \dot{a}^3 - \frac{8\pi G}{3}\frac{d(\rho_m a^3)}{dt} = \frac{-Kc^2}{r_0^2}\dot{a}$ (18)

We know that $\frac{d(\rho_e a^3)}{dt} + \frac{w\rho_e\, d(a^3)}{dt} = 0$, but $\rho_e = \rho_m c^2$ so $\frac{d(\rho_m c^2 a^3)}{dt} + \frac{w\rho_m c^2\, d(a^3)}{dt} = 0 \Rightarrow \frac{d(\rho_m a^3)}{dt} + \frac{w\rho_m\, d(a^3)}{dt} = 0$

using $\frac{d(\rho_m a^3)}{dt} + \frac{w\rho_m\, d(a^3)}{dt} = 0$ in (18), we obtain

$2a\,\dot{a}\,\ddot{a} + \dot{a}^3 + \frac{8\pi G}{3}w\rho_m\frac{d(a^3)}{dt} = \frac{-Kc^2}{r_0^2}\dot{a}$, but we know that $K = -\left(\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G\rho_m}{3}\right)\frac{a^2}{c^2}r_0^2$, so it becomes:

$2a\,\dot{a}\,\ddot{a} + \dot{a}^3 + 8\pi G w\rho_m a^2\dot{a} = \dot{a}^3 - \frac{8\pi G\rho_m}{3}a^2\dot{a}$.

By divinding with $2a\dot{a}$, we obtain the acceleration $\ddot{a}$:

$\ddot{a} = -4\pi G\rho_m a\left(w + \frac{1}{3}\right)$ (19)

Knowing that $\rho_m = \rho_{0,m}a^{-3(1+w)}$, we have $a = \left(\frac{\rho_m}{\rho_{0,m}}\right)^{\frac{-1}{3(1+w)}}$ (20)

Substituting in the last equation, we obtain the final result

$\ddot{a} = -4\pi G\rho_{0,m}^{\frac{1}{3(1+w)}}\rho_m^{\frac{2-3w}{3(1+w)}}\left(w + \frac{1}{3}\right)$ (21)

We observe that both ρ_m and $\rho_{0,m}$ are positive. From this observation we conclude that the sign of the acceleration only depends on the sign of $w+\frac{1}{3}$.

If $w>-\frac{1}{3}$ then the expansion of the universe will decelerate.

If $w<-\frac{1}{3}$ then the expansion of the universe will accelerate.

Finally, if $w=-\frac{1}{3}$ then the universe will expand with constant velocity.

Scale factor as a function of time

From (19), substituting ρ_m from (20) we have:

$\ddot{a}=-4\pi G\rho_{0,m}a^{-3(1+w)+1}\left(w+\frac{1}{3}\right)$

For any $w\neq-\frac{1}{3}$ we have :

$d(\dot{a}^2)=2\ddot{a}da=-8\pi G\rho_{0,m}\left(w+\frac{1}{3}\right)a^{-2-3w}da$

$\dot{a}^2=C_0-8\pi G\rho_{0,m}\left(w+\frac{1}{3}\right)\frac{a^{-1-3w}}{-1-3w}=C_0+\frac{8\pi G\rho_{0,m}}{3}a^{-1-3w}$

$\dot{a}=\sqrt{C_0+\frac{8\pi G\rho_{0,m}}{3}a^{-1-3w}}$ (22)

$\frac{da}{\sqrt{C_0+\frac{8\pi G\rho_{0,m}}{3}a^{-1-3w}}}=dt$ (23)

A first observation is that $\frac{1}{2}mC_0=\frac{1}{2}m\dot{a}^2-\frac{4\pi Gm\rho_{0,m}}{3}a^{-1-3w}=\frac{E_{m,0}}{r_0^2}$, where $E_{m,0}$ is the total energy of a particle of mass m in the universe.

A second observation is that there is no function in closed form whose differential is the left term in (22).

Thus, we will consider 3 cases:

I) $C_0=0$, which means that the total energy of any particle in the universe is 0 ,and also that K from 2.1) is equal to 0 .

Here, (23) becomes

$\sqrt{\frac{3}{8\pi G\rho_{0,m}}}a^{\frac{1+3w}{2}}da=dt$

which can be easily integrated (on two cases):

a) $w\neq-1$

$\sqrt{\frac{3}{8\pi G\rho_{0,m}}}\frac{a(t)^{\frac{3(1+w)}{2}}-a(t_0)^{\frac{3(1+w)}{2}}}{\frac{3(1+w)}{2}}=t-t_0$

$\sqrt{\frac{1}{6\pi G\rho_{0,m}}}\left(a(t)^{\frac{3(1+w)}{2}}-1\right)=t-t_0$

$a^{\frac{3(1+w)}{2}}-1=\sqrt{6\pi G\rho_{0,m}}(t-t_0)$

$a(t)=\left(1+\sqrt{6\pi G\rho_{0,m}}(t-t_0)\right)^{\frac{2}{3(1+w)}}$ (24)

Considering that the universe had a Big Bang (that would mean that $a_{(0)}=0$) .

$a(0)=0\Rightarrow1-\sqrt{6\pi G\rho_{0,m}}t_0=0\Rightarrow a(t)={}^{3(1+w)}\sqrt{6\pi G\rho_{0,m}}t^{\frac{2}{3(1+w)}}$ (24')

From (14'), we have $\rho_m=\rho_{0,m}\frac{1}{6\pi G\rho_{0,m}t^2}=\frac{1}{6\pi Gt^2}$, which is true for any w (except -1, of course)

For matter, $a(t) = \sqrt[3]{6\pi G\rho_{0,m}}\, t^{\frac{2}{3}}$

For radiation, $a(t) = \sqrt[4]{6\pi G\rho_{0,m}}\, t^{\frac{1}{2}}$

From thermodynamics, we know that $\frac{uc}{4} = \sigma T^4$ [1], so $T = \sqrt[4]{\frac{\rho_m c}{4\sigma}} = \frac{1}{k}\sqrt[4]{\frac{5c^3h^3}{16\pi^5 G}}\sqrt{\frac{1}{t}}$, where k is the Boltzmann constant , h is the Planck constant , and $\sigma = \frac{2\pi^5 k^4}{15c^2h^3}$ is the Stefan-Boltzmann constant .

b) $w = -1$

$\sqrt{\frac{3}{8\pi G\rho_{0,\mathrm{de}}}}\frac{da}{a} = dt$

$\sqrt{\frac{3}{8\pi G\rho_{0,\mathrm{de}}}}\ln\left(\frac{a(t)}{a(t_0)}\right) = \sqrt{\frac{3}{8\pi G\rho_{0,\mathrm{de}}}}\ln(a(t)) = t - t_0$

$a(t) = e^{\sqrt{\frac{8\pi G\rho_{0,\mathrm{de}}}{3}}(t-t_0)} = e^{H_0(t-t_0)}$ (24")

$\rho(t) = \rho_{0,\mathrm{de}}$

II) $C_0 = \frac{-c^2}{r_0^2}$(that means that the K from 2.1) is equal to 1, and also that the energy of the universe is less than 0) :

$$\frac{da}{\sqrt{\frac{-c^2}{r_0^2} + \frac{8\pi G\rho_{0,m}}{3}a^{-1-3w}}} = dt \Rightarrow \frac{da}{\sqrt{-1 + \frac{8\pi G\rho_{0,m}}{3\frac{c^2}{r_0^2}}a^{-1-3w}}} = \frac{c}{r_0}dt$$

We shall define $\frac{8\pi G\rho_{0,m}}{3\frac{c^2}{r_0^2}} = B^{-1-3w} = \text{constant}$, to simplify our calculations .

$$\frac{da}{\sqrt{-1 + (\mathrm{Ba})^{-1-3w}}} = \frac{c}{r_0}dt \Rightarrow \frac{d(\mathrm{Ba})}{\sqrt{-1 + (\mathrm{Ba})^{-1-3w}}} = \frac{\mathrm{cB}}{r_0}dt \Rightarrow \frac{d(\mathrm{Ba})}{\sqrt{(\mathrm{Ba})^{-1-3w} - 1}} = \frac{\mathrm{cB}}{r_0}dt \Rightarrow \frac{d(\mathrm{Ba})}{\sqrt{\frac{1}{(\mathrm{Ba})^{1+3w}} - 1}} = \frac{\mathrm{cB}}{r_0}dt$$

The integral on the left side doesn't have a closed form .

We shall treat the 3 particular cases of interest:

a) For matter ($w = 0$) : $\frac{d(\mathrm{Ba})}{\sqrt{\frac{1}{(\mathrm{Ba})} - 1}} = \frac{\mathrm{cB}}{r_0}dt$

We can compute the integral on the left side using a trigonometric substitution : $\mathrm{Ba} = \sin^2(\alpha) \Rightarrow$

$$\frac{d(\sin^2(\alpha))}{\sqrt{\frac{1}{\sin^2(\alpha)} - 1}} = \frac{\mathrm{cB}}{r_0}dt \Rightarrow \frac{2\sin(\alpha)\, d(\sin(\alpha))}{\sqrt{\frac{1-\sin^2(\alpha)}{\sin^2(\alpha)}}} = \frac{\mathrm{cB}}{r_0}dt \Rightarrow \frac{2\sin(\alpha)\cos(\alpha)\, d\alpha}{\frac{\cos(\alpha)}{\sin(\alpha)}} = \frac{\mathrm{cB}}{r_0}dt \Rightarrow 2\sin^2(\alpha)\, d\alpha = \frac{\mathrm{cB}}{r_0}dt$$

$$\Rightarrow 2\sin^2(\alpha)\, d\alpha = 2\left(\frac{1-\cos(2\alpha)}{2}\right)d\alpha = d\alpha - \cos(2\alpha)\frac{d(2\alpha)}{2} = d\alpha - \frac{d(\sin(2\alpha))}{2}$$

$\Rightarrow t = \frac{r_0}{\mathrm{cB}}\left(\alpha - \alpha_{(0)} - \frac{\sin(2\alpha) - \sin(2\alpha_{(0)})}{2}\right)$, where $\alpha_{(0)} = \arcsin\left(\sqrt{\mathrm{Ba}_{(0)}}\right) = 0$, assuming that this universe had a Big Bang ($a_{(0)} = 0$).

$$\Rightarrow t = \frac{r_0}{\mathrm{cB}}\left(\alpha - \frac{\sin(2\alpha)}{2}\right) = \frac{r_0\frac{8\pi G\rho_{0,m}}{3\frac{c^2}{r_0^2}}}{c}\left(\alpha - \frac{\sin(2\alpha)}{2}\right) = \frac{8\pi G\rho_{0,m}r_0^3}{3c^3}\left(\alpha - \frac{\sin(2\alpha)}{2}\right)$$

$$\Rightarrow \begin{cases} \boldsymbol{t = \frac{8\pi G\rho_{0,m}r_0^3}{3c^3}\left(\alpha - \frac{\sin(2\alpha)}{2}\right)} \\ \boldsymbol{a = \frac{8\pi G\rho_{0,m}r_0^2}{3c^2}\left(\frac{1}{2} - \frac{\cos(2\alpha)}{2}\right)} \end{cases}$$

We cannot have a explicit function for $a_{(t)}$ because for that we need to find $\alpha(\mathrm{t})$ which would imply to solve a transcendent equation , that being impossible through analytic methods.

1. *http://www.pha.jhu.edu/~kknizhni/StatMech/Derivation_of_Stefan_Boltzmann_Law.pdf*

We can find $\rho_{m(\alpha)}=\rho_{0,m}\left(\frac{8\pi G\rho_{0,m}r_0^2}{3c^2}\left(\frac{1}{2}-\frac{\cos(2\alpha)}{2}\right)\right)^{-3}$

b) For radiation ($w=\frac{1}{3}$) :

$$\frac{d(\mathrm{Ba})}{\sqrt{\frac{1}{(\mathrm{Ba})^2}-1}}=\frac{\mathrm{cB}}{r_0}dt\Rightarrow\frac{d(\mathrm{Ba})}{\sqrt{\frac{1}{(\mathrm{Ba})^2}-1}}=\frac{d(\mathrm{Ba})}{\sqrt{\frac{1-(\mathrm{Ba})^2}{(\mathrm{Ba})^2}}}=\frac{d(\mathrm{Ba})\,\mathrm{Ba}}{\sqrt{1-(\mathrm{Ba})^2}}=\frac{d((\mathrm{Ba})^2)}{2\sqrt{1-(\mathrm{Ba})^2}}=-\frac{d(1-(\mathrm{Ba})^2)}{2\sqrt{1-(\mathrm{Ba})^2}}=-d\left(\sqrt{1-(\mathrm{Ba})^2}\right)$$

$\Rightarrow d\left(\sqrt{1-(\mathrm{Ba})^2}\right)=-\frac{\mathrm{cB}}{r_0}dt\Rightarrow\sqrt{1-(\mathrm{Ba})^2}-\sqrt{1-(\mathrm{Ba}_{(0)})^2}=-\frac{\mathrm{cB}}{r_0}t$, where $a_{(0)}=0$, assuming that this universe had a Big Bang.

$$\sqrt{1-(\mathrm{Ba})^2}=1-\frac{\mathrm{cB}}{r_0}t\Rightarrow 1-(\mathrm{Ba})^2=\left(1-\frac{\mathrm{cB}}{r_0}t\right)^2\Rightarrow(\mathrm{Ba})^2=1-\left(1-\frac{\mathrm{cB}}{r_0}t\right)^2\Rightarrow\mathrm{Ba}=\sqrt{1-\left(1-\frac{\mathrm{cB}}{r_0}t\right)^2}$$

$$a_{(t)}=\left(\frac{3\frac{c^2}{r_0^2}}{8\pi G\rho_{0,r}}\right)^{-1/2}\left(1-\left(1-\frac{c\left(\frac{3\frac{c^2}{r_0^2}}{8\pi G\rho_{0,r}}\right)^{1/2}}{r_0}t\right)^2\right)^{1/2}$$

$$\Rightarrow\rho_{r(t)}=\rho_{0,r}\left(\frac{\sqrt{1-\left(1-\frac{\mathrm{cB}}{r_0}t\right)^2}}{B}\right)^{-4}=\rho_{0,r}\left(\left(\frac{3\frac{c^2}{r_0^2}}{8\pi G\rho_{0,r}}\right)^{-1/2}\left(1-\left(1-\frac{c\left(\frac{3\frac{c^2}{r_0^2}}{8\pi G\rho_{0,\mathrm{mr}}}\right)^{1/2}}{r_0}t\right)^2\right)^{1/2}\right)^{-4}$$

For this case we can also calculate the time dependance of the temperature of the radiation :

$$\frac{uc}{4}=\sigma T^4\Rightarrow T=\left(\frac{\rho_{r(t)}c}{4\sigma}\right)^{1/4}=\left(\frac{c}{4\sigma}\rho_{0,r}\left(\frac{\sqrt{1-\left(1-\frac{\mathrm{cB}}{r_0}t\right)^2}}{B}\right)^{-4}\right)^{1/4}\text{ where }B=\left(\frac{8\pi G\rho_{0,r}}{3\frac{c^2}{r_0^2}}\right)^{-1/2}$$

c) For dark energy ($w=-1$) :

$$\frac{d(\mathrm{Ba})}{\sqrt{\frac{1}{(\mathrm{Ba})^{-2}}-1}}=\frac{\mathrm{cB}}{r_0}dt\Rightarrow\frac{d(\mathrm{Ba})}{\sqrt{(\mathrm{Ba})^2-1}}=\frac{\mathrm{cB}}{r_0}dt$$

Using Wolfram Alpha we can compute the integral on the left side :

$$\ln\left(\frac{\sqrt{(\mathrm{Ba})^2-1}+\mathrm{Ba}}{\sqrt{(\mathrm{Ba}_{(0)})^2-1}+\mathrm{Ba}_{(0)}}\right)=\frac{\mathrm{cB}}{r_0}t\Rightarrow\sqrt{(\mathrm{Ba})^2-1}+\mathrm{Ba}=\left(\sqrt{(\mathrm{Ba}_{(0)})^2-1}+\mathrm{Ba}_{(0)}\right)e^{\frac{\mathrm{cB}}{r_0}t}$$

$$\Rightarrow(\mathrm{Ba})^2-1=(\mathrm{Ba})^2-2\mathrm{Ba}\left(\sqrt{(\mathrm{Ba}_{(0)})^2-1}+\mathrm{Ba}_{(0)}\right)e^{\frac{\mathrm{cB}}{r_0}t}+\left(\left(\sqrt{(\mathrm{Ba}_{(0)})^2-1}+\mathrm{Ba}_{(0)}\right)e^{\frac{\mathrm{cB}}{r_0}t}\right)^2$$

$$\Rightarrow 2\mathrm{Ba}\left(\sqrt{(\mathrm{Ba}_{(0)})^2-1}+\mathrm{Ba}_{(0)}\right)e^{\frac{\mathrm{cB}}{r_0}t}=\left(\left(\sqrt{(\mathrm{Ba}_{(0)})^2-1}+\mathrm{Ba}_{(0)}\right)e^{\frac{\mathrm{cB}}{r_0}t}\right)^2+1$$

$$\Rightarrow a=\frac{\left(\sqrt{(\mathrm{Ba}_{(0)})^2-1}+\mathrm{Ba}_{(0)}\right)e^{\frac{\mathrm{cB}}{r_0}t}+\left(\sqrt{(\mathrm{Ba}_{(0)})^2-1}+\mathrm{Ba}_{(0)}\right)^{-1}e^{\frac{-\mathrm{cB}}{r_0}t}}{2B}$$

This universe behaves now in a very different way than before . It seems that if this universe had a Big Bang($a_{(0)}=0$) , then $a_{(t)}$ would somehow be imaginary . That , at least in our current understanding , is not possible. So that would mean that the initial assumption is incorrect . This kind of universe did not have a Big Bang ($a_{(0)}=0$) . We shall take the first value of a_0 that will not give us a imaginary value for $a_{(t)}$.

That would be $a_0=\frac{1}{B}\Rightarrow a_{(t)}=\frac{e^{\frac{\mathrm{cB}}{r_0}t}+e^{\frac{-\mathrm{cB}}{r_0}t}}{2B}=\frac{\sinh\left(\frac{\mathrm{cB}}{r_0}t\right)}{B}$, where $B=\left(\frac{8\pi G\rho_{0,\mathrm{de}}}{3\frac{c^2}{r_0^2}}\right)^{1/2}$

$\rho_{\mathrm{de}}=\rho_{0,\mathrm{de}}=$constant

III) $C_0 = \frac{c^2}{r_0^2}$(that means that the K from 2.1) is equal to -1, and also that the energy of the universe is bigger than 0) :

$$\frac{da}{\sqrt{\frac{c^2}{r_0^2} + \frac{8\pi G\rho_{0,m}}{3} a^{-1-3w}}} = dt \Rightarrow \frac{da}{\sqrt{1 + \frac{8\pi G\rho_{0,m}}{3\frac{c^2}{r_0^2}} a^{-1-3w}}} = \frac{c}{r_0} dt$$

We shall define $\frac{8\pi G\rho_{0,m}}{3\frac{c^2}{r_0^2}} = B^{-1-3w} = \text{constant}$, to simplify our calculations .

$$\frac{da}{\sqrt{1 + (\mathrm{Ba})^{-1-3w}}} = \frac{c}{r_0} dt \Rightarrow \frac{d(\mathrm{Ba})}{\sqrt{1 + (\mathrm{Ba})^{-1-3w}}} = \frac{\mathrm{cB}}{r_0} dt \Rightarrow \frac{d(\mathrm{Ba})}{\sqrt{1 + \frac{1}{(\mathrm{Ba})^{1+3w}}}} = \frac{\mathrm{cB}}{r_0} dt$$

The integral on the left side doesn‘t have a closed form .

We shall treat the 3 particular cases of interest:

a) For matter ($w = 0$) :

$$\frac{d(\mathrm{Ba})}{\sqrt{1 + \frac{1}{(\mathrm{Ba})}}} = \frac{\mathrm{cB}}{r_0} dt$$

We can compute the integral on the left side using a hyperbolic substitution : $\mathrm{Ba} = \sinh^2(\alpha)$

$$\Rightarrow \frac{d(\sinh^2(\alpha))}{\sqrt{1 + \frac{1}{\sinh^2(\alpha)}}} = \frac{\mathrm{cB}}{r_0} dt \Rightarrow \frac{2\sinh(\alpha)\cosh(\alpha)\, d\alpha}{\frac{\cosh(\alpha)}{\sinh(\alpha)}} = \frac{\mathrm{cB}}{r_0} dt \Rightarrow 2\sinh^2(\alpha)\, d\alpha = \frac{\mathrm{cB}}{r_0} dt \Rightarrow (\cosh(2\alpha) - 1)\, d\alpha = \frac{\mathrm{cB}}{r_0} dt$$

$\Rightarrow \cosh(2\alpha)\frac{d(2\alpha)}{2} - d\alpha = \frac{\mathrm{cB}}{r_0} dt \Rightarrow t = \frac{r_0}{\mathrm{cB}}\left(\frac{\sinh(2\alpha) - \sinh(2\alpha_{(0)})}{2} - (\alpha - \alpha_{(0)})\right)$, where $\alpha_{(0)} = \operatorname{argsinh}(\sqrt{\mathrm{Ba}_0})$, where

$\operatorname{argsinh}(x)$ is the argument of the hyperbolic function $\sinh(x)$. $a_{(0)} = 0$, assuming that this universe had a Big Bang . $\Rightarrow \alpha_0 = 0 \Rightarrow t = \frac{r_0}{\mathrm{cB}}\left(\frac{\sinh(2\alpha)}{2} - \alpha\right)$

Also $a_{(\alpha)} = \frac{(\cosh(2\alpha) - 1)}{2B}$.

We cannot have a explicit function for $a_{(t)}$ because for that we need to find $\alpha(\mathrm{t})$ which would imply to solve a transcendent equation , that being impossible through analytic methods.

We can find $\rho_{m(\alpha)} = \rho_{0,m}\left(\frac{(\cosh(2\alpha) - 1)}{2B}\right)^{-3}$, where $B = \left(\frac{8\pi G\rho_{0,m}}{3\frac{c^2}{r_0^2}}\right)^{-1}$

b) For radiation ($w = \frac{1}{3}$)

$$\frac{d(\mathrm{Ba})}{\sqrt{\frac{1}{(\mathrm{Ba})^2} + 1}} = \frac{\mathrm{cB}}{r_0} dt \Rightarrow \frac{d(\mathrm{Ba})}{\sqrt{\frac{1}{(\mathrm{Ba})^2} + 1}} = \frac{d(\mathrm{Ba})}{\sqrt{\frac{1 + (\mathrm{Ba})^2}{(\mathrm{Ba})^2}}} = \frac{d(\mathrm{Ba})\,\mathrm{Ba}}{\sqrt{1 + (\mathrm{Ba})^2}} = \frac{d((\mathrm{Ba})^2)}{2\sqrt{1 + (\mathrm{Ba})^2}} = \frac{d(1 + (\mathrm{Ba})^2)}{2\sqrt{1 + (\mathrm{Ba})^2}} = d\left(\sqrt{1 + (\mathrm{Ba})^2}\right)$$

$\Rightarrow d\left(\sqrt{1 + (\mathrm{Ba})^2}\right) = \frac{\mathrm{cB}}{r_0} dt \Rightarrow \sqrt{1 + (\mathrm{Ba})^2} - \sqrt{1 + (\mathrm{Ba}_{(0)})^2} = \frac{\mathrm{cB}}{r_0} t$, where $a_{(0)} = 0$, assuming that this universe had a Big Bang.

$$\Rightarrow \sqrt{1 + (\mathrm{Ba})^2} - 1 = \frac{\mathrm{cB}}{r_0} t \Rightarrow 1 + (\mathrm{Ba})^2 = \left(1 + \frac{\mathrm{cB}}{r_0} t\right)^2 \Rightarrow \mathrm{Ba} = \sqrt{\left(1 + \frac{\mathrm{cB}}{r_0} t\right)^2 - 1}$$

$\Rightarrow a = \frac{1}{B}\sqrt{\left(1 + \frac{\mathrm{cB}}{r_0} t\right)^2 - 1}$, where $\frac{8\pi G\rho_{0,m}}{3\frac{c^2}{r_0^2}} = B^{-2} \Rightarrow B = \left(\frac{8\pi G\rho_{0,r}}{3\frac{c^2}{r_0^2}}\right)^{-1/2}$

$$\Rightarrow \rho_{r(t)} = \rho_{0,r}\left(\frac{\sqrt{\left(1 + \frac{\mathrm{cB}}{r_0} t\right)^2 - 1}}{B}\right)^{-4}$$

For this case we can also calculate the time dependance of the temperature of the radiation :

$$\frac{uc}{4}=\sigma T^4 \Rightarrow T=\left(\frac{\rho_{r(t)}c}{4\sigma}\right)^{1/4}=\left(\frac{c}{4\sigma}\rho_{0,r}\left(\frac{\sqrt{\left(1+\frac{\mathrm{cB}}{r_0}t\right)^2-1}}{B}\right)^{-4}\right)^{1/4}$$

c)For dark energy ($w=-1$) :

$$\frac{d(\mathrm{Ba})}{\sqrt{\frac{1}{(\mathrm{Ba})^{-2}}+1}}=\frac{\mathrm{cB}}{r_0}dt \Rightarrow \frac{d(\mathrm{Ba})}{\sqrt{(\mathrm{Ba})^2+1}}=\frac{\mathrm{cB}}{r_0}dt$$

Using Wolfram Alpha we can compute the integral on the left side :

$$\ln\left(\frac{\sqrt{(\mathrm{Ba})^2+1}+\mathrm{Ba}}{\sqrt{(\mathrm{Ba}_{(0)})^2+1}+\mathrm{Ba}_{(0)}}\right)=\frac{\mathrm{cB}}{r_0}t \Rightarrow \sqrt{(\mathrm{Ba})^2+1}+\mathrm{Ba}=\left(\sqrt{(\mathrm{Ba}_{(0)})^2+1}+\mathrm{Ba}_{(0)}\right)e^{\frac{\mathrm{cB}}{r_0}t}$$

$$\Rightarrow (\mathrm{Ba})^2+1=(\mathrm{Ba})^2-2\mathrm{Ba}\left(\sqrt{(\mathrm{Ba}_{(0)})^2+1}+\mathrm{Ba}_{(0)}\right)e^{\frac{\mathrm{cB}}{r_0}t}+\left(\left(\sqrt{(\mathrm{Ba}_{(0)})^2+1}+\mathrm{Ba}_{(0)}\right)e^{\frac{\mathrm{cB}}{r_0}t}\right)^2$$

$$\Rightarrow 2\mathrm{Ba}\left(\sqrt{(\mathrm{Ba}_{(0)})^2+1}+\mathrm{Ba}_{(0)}\right)e^{\frac{\mathrm{cB}}{r_0}t}=\left(\left(\sqrt{(\mathrm{Ba}_{(0)})^2+1}+\mathrm{Ba}_{(0)}\right)e^{\frac{\mathrm{cB}}{r_0}t}\right)^2-1$$

$$\Rightarrow a=\frac{\left(\sqrt{(\mathrm{Ba}_{(0)})^2+1}+\mathrm{Ba}_{(0)}\right)e^{\frac{\mathrm{cB}}{r_0}t}+\left(\sqrt{(\mathrm{Ba}_{(0)})^2+1}+\mathrm{Ba}_{(0)}\right)^{-1}e^{\frac{-\mathrm{cB}}{r_0}t}}{2B}$$

If we assume that the universe had a Big Bang $\Rightarrow a_{(0)}=0$

$$\Rightarrow a_{(t)}=\frac{e^{\frac{\mathrm{cB}}{r_0}t}+e^{\frac{-\mathrm{cB}}{r_0}t}}{2B}=\frac{\cosh\left(\frac{\mathrm{cB}}{r_0}t\right)}{B}$$

,where $B=\left(\frac{8\pi G\rho_{0,\mathrm{de}}}{3\frac{c^2}{r_0^2}}\right)^{1/2}$

$\rho_{\mathrm{de}}=\rho_{0,\mathrm{de}}=$constant

We shall now consider the case when $w=-\frac{1}{3}$:

$\ddot{a}=-4\pi G\rho_{0,m}a^{-3\left(1-\frac{1}{3}\right)+1}\left(-\frac{1}{3}+\frac{1}{3}\right)=0 \Rightarrow \dot{a}^2=C_0=\frac{-\mathrm{Kc}^2}{r_0^2}$, where K is the K used and defined at 2.1) .

If $K=1 \Rightarrow \dot{a}^2=\frac{-c^2}{r_0^2} \Rightarrow \dot{a}$ is imaginary , which is impossible , so a universe like this can not exist .

If $K=0 \Rightarrow \dot{a}^2=0 \Rightarrow a=a_{(0)}=\text{constant} \Rightarrow \rho_m=\rho_{0,m}a_{(0)}^{-2}$

If $K=-1 \Rightarrow \dot{a}^2=\frac{c^2}{r_0^2} \Rightarrow a-a_{(0)}=\pm\frac{c}{r_0}t \Rightarrow a_{(t)}=a_{(0)}\pm\frac{c}{r_0}t \Rightarrow \rho_m=\rho_{0,m}\left(a_{(0)}\pm\frac{c}{r_0}t\right)^{-2}$

In a lot of the cases described above we have used the fact that if the universe we were talking about had a "Big Bang" , then $a_{(0)}=0$, so the size of that universe at its beginning was 0 . What we mean by "Big Bang" , is the fact that the universe we were talking about had a beginning similar to ours . That is , it expended from "nothingness", a quantum fluctuation, a singularity , or in more mathematical terms $a_{(0)}=0$.

To find if a universe will collapse or expand forever we need to take the differantial equation:

$\ddot{a}=-4\pi G\rho_{0,m}a^{-(2+3w)}\left(w+\frac{1}{3}\right)$

and solve it to find $a_{(t)}$. If the equation $a_{(t)}=0$ has any other solution besides the trivial one $a_{(0)}=0$ then the univers will collapse . If not , the universe will expand forever.

We shall now make a similar analysis to what we did at 2.6) .

If $w\neq\frac{1}{3}$:

I) If K=0 (the energy of the universe is equal to 0) :

a) If $w \neq -1$:

$a_{(t)} = {}^{3(1+w)}\sqrt{6\pi G\rho_{0,m}} t^{\frac{2}{3(1+w)}} = 0 \Rightarrow t = 0 \Rightarrow a_{(0)} = 0$

In this case the only solution is the trivial one so for any w , including matter and radiation with the conditions above the universe will expand forever.

b) If $w = -1$:

$a_{(t)} = e^{\sqrt{\frac{8\pi G\rho_{0,m}}{3}}(t-t_0)} = 0 \Rightarrow$ There are no solutions to this equation, so this universe will expand forever.

II) If $K = 1$ (the energy of the universe is less than 0) :

$$\frac{d(\mathrm{Ba})}{\sqrt{\frac{1}{(\mathrm{Ba})^{1+3w}} - 1}} = \frac{\mathrm{cB}}{r_0} dt$$

For our universe (multiple fluids)

It has been determined that our universe is flat with a very small (0.4%) margin of error[2].

So, it's energy is $\simeq 0$ and $K = 0$.

In the Friedmann equation derivation at (2.1) ρ_m is the total equivalent mass density (the sum of the 3 mass and equivalent mass density).

$$\left(H^2 - \frac{8\pi G\rho_m}{3}\right)a^2 = \frac{-\mathrm{Kc}^2}{r_0^2} = 0$$

$\left(H^2 - \frac{8\pi G(\rho_b + \rho_r + \rho_{de})}{3}\right)a^2 = 0$, where ρ_b is the baryonic mass density, ρ_r is the equivalent photon mass density and ρ_{de} is the equivalent dark energy mass density.

Now we shall use (14'):

$$\rho_b = \rho_{b,0}a^{-3(1+w_b)} \qquad \rho_r = \rho_{r,0}a^{-3(1+w_r)} \qquad \rho_{de} = \rho_{de,0}a^{-3(1+w_{de})} \quad (25)$$

$$\rho_0 = \rho_{b,0} + \rho_{r,0} + \rho_{de,0} \quad (26)$$

$$\Omega_{b,0} = \frac{\rho_{b,0}}{\rho_0} = 0.27 \pm 0.04\% \qquad \Omega_{r,0} = \frac{\rho_{r,0}}{\rho_0} = 8.24 \times 10^{-5} \qquad \Omega_{de,0} = \frac{\rho_{de,0}}{\rho_0} = 0.73 \pm 0.04\% \quad (27)$$

$$\left(H^2 - \frac{8\pi G\rho_0(\Omega_{b,0}a^{-3(1+w_b)} + \Omega_{r,0}a^{-3(1+w_r)} + \Omega_{de,0}a^{-3(1+w_{de})})}{3}\right)a^2 = 0$$

As we can see from the numerical values, the radiation component is negligible, so we can write:

$$\left(H^2 - \frac{8\pi G\rho_0(\Omega_{b,0}a^{-3(1+w_b)} + \Omega_{de,0}a^{-3(1+w_{de})})}{3}\right)a^2 = 0$$

Substituting H_0 with $\frac{\dot{a}}{a}$ we obtain:

$\dot{a}^2 = \frac{8\pi G\rho_0}{3}\left(\Omega_{b,0}a^{-(1+3w_b)} + \Omega_{de,0}a^{-(1+3w_{de})}\right)$, where $w_b = 0$ and $w_{de} = -1$

$$\dot{a}^2 = \frac{8\pi G\rho_0}{3}(\Omega_{b,0}a^{-1} + \Omega_{de,0}a^2)$$

$$\dot{a} = \sqrt{\frac{8\pi G\rho_0}{3}(\Omega_{b,0}a^{-1} + \Omega_{de,0}a^2)}$$

Writing it in a differential form:

$$\frac{da}{\sqrt{\frac{8\pi G\rho_0}{3}(\Omega_{b,0}a^{-1} + \Omega_{de,0}a^2)}} = dt$$

2. http://map.gsfc.nasa.gov/universe/uni_shape.html

$$\frac{da}{\sqrt{\Omega_{b,0}a^{-1}+\Omega_{de,0}a^2}}=\sqrt{\frac{8\pi G\rho_0}{3}}dt$$

$$\frac{\sqrt{a}\,da}{\sqrt{1+\frac{\Omega_{de,0}}{\Omega_{b,0}}a^3}}=\sqrt{\Omega_{b,0}\frac{8\pi G\rho_0}{3}}t=H_0\sqrt{\Omega_{b,0}}dt \ (28)$$

Integrating the left and right parts between 0 and a, respectively 0 and t, we have:

$$\frac{2\text{argsh}\left(\sqrt{\frac{\Omega_{de,0}}{\Omega_{b,0}}}a^{\frac{3}{2}}\right)}{3\sqrt{\frac{\Omega_{de,0}}{\Omega_{b,0}}}}=H_0\sqrt{\Omega_{b,0}}t \ (29)$$

$$\text{argsh}\left(\sqrt{\frac{\Omega_{de,0}}{\Omega_{b,0}}}a^{\frac{3}{2}}\right)=\frac{3}{2}H_0\sqrt{\Omega_{de,0}}t$$

And here we find $a(t)$.

$$a(t)=\sqrt[3]{\frac{\Omega_{b,0}}{\Omega_{de,0}}}\sinh^{\frac{2}{3}}\left(\frac{3}{2}H_o t\sqrt{\Omega_{de,0}}\right) \ (30)$$

From (25):

$$\rho_b(t)=\rho_0\Omega_{b,0}a^{-3}=\rho_0\Omega_{de,0}\sinh^{-2}\left(\frac{3}{2}H_0\sqrt{\Omega_{de,0}}t\right) \ (31)$$

$$\rho_r(t)=\rho_0\Omega_{r,0}a^{-4}=\rho_0\Omega_{r,0}\left(\frac{\Omega_{de,0}}{\Omega_{b,0}}\right)^{\frac{4}{3}}\sinh^{-\frac{8}{3}}\left(\frac{3}{2}H_o\sqrt{\Omega_{de,0}}t\right) \ (32)$$

$$\rho_{de}(t)=\rho_0\Omega_{de,0}a^0=\rho_0\Omega_{\text{de},0} \ (33)$$

$$\rho(t)=\rho_0\left(\Omega_{de,0}\sinh^{-2}\left(\frac{3}{2}H_0\sqrt{\Omega_{de,0}}t\right)+\Omega_{r,0}\left(\frac{\Omega_{de,0}}{\Omega_{b,0}}\right)^{\frac{4}{3}}\sinh^{-\frac{8}{3}}\left(\frac{3}{2}H_o\sqrt{\Omega_{de,0}}t\right)+\Omega_{de,0}\right) \ (34)$$

If we neglect the radiation part:

$$\rho(t)=\rho_0\left(\Omega_{de,0}\sinh^{-2}\left(\frac{3}{2}H_0\sqrt{\Omega_{de,0}}t\right)+\Omega_{de,0}\right)=\rho_0\Omega_{de,0}\left(1+\sinh^{-2}\left(\frac{3}{2}H_0\sqrt{\Omega_{de,0}}t\right)\right) \ (34')$$

Age of the universe, periods of domination

For finding the age of the universe, we will equate $\rho(t_0)$ with ρ_0.

$$\rho(t_0)=\rho_0\Rightarrow 1=\Omega_{de,0}\left(1+\sinh^{-2}\left(\frac{3}{2}H_0\sqrt{\Omega_{de,0}}t\right)\right)\Rightarrow\sinh^{-2}\left(\frac{3}{2}H_0\sqrt{\Omega_{de,0}}t_0\right)=\frac{1}{\Omega_{de,0}}-1$$

$$\frac{3}{2}H_0\sqrt{\Omega_{de,0}}t_0=\text{argsh}\left(\sqrt{\frac{\Omega_{de,0}}{1-\Omega_{de,0}}}\right)$$

$$t_0=\frac{2}{3}\frac{\text{argsh}\left(\sqrt{\frac{\Omega_{de,0}}{1-\Omega_{de,0}}}\right)}{H_0\sqrt{\Omega_{de,0}}} \ (35)$$

For $H_0=71.7\,\frac{\text{km}}{\text{s}}$, $t_0=13.54\times10^9$ years

For $H_0=67.8\,\frac{\text{km}}{\text{s}}$ [3], $t_0=14.32\times10^9$ years

For finding the periods of domination for each fluid type, we will compare their densities. The fluid with the largest density will be the dominant one in the same volume.

From (31) and (32):

$$\rho_b>\rho_r\Rightarrow\Omega_{b,0}\,a^{-3}>\Omega_{r,0}\,a^{-4}\Rightarrow a>\frac{\Omega_{r,0}}{\Omega_{b,0}}$$

From (30):

3. http://arxiv.org/abs/1303.5062

$\sqrt[3]{\frac{\Omega_{b,0}}{\Omega_{de,0}}}\sinh^{\frac{2}{3}}(\frac{3}{2}H_o t\sqrt{\Omega_{de,0}}) > \frac{\Omega_{r,0}}{\Omega_{b,0}}$

Plugging numerical values in:

$0.7178\sinh^{\frac{2}{3}}(1.2816H_0 t) > 3.05\times10^{-4} \Rightarrow t > t_1 = 21366 \approx 9.37\times10^5$ years

From (32) and (33):

$\rho_r > \rho_{de} \Rightarrow \Omega_{r,0}a^{-4} > \Omega_{de,0} \Rightarrow a < \sqrt[4]{\frac{\Omega_{r,0}}{\Omega_{de,0}}}$

From (30) and by plugging numerical values in, we have $t < t_2 = 5.8\times10^8$ years

From (31) and (33):

$\rho_b > \rho_{de} \Rightarrow \Omega_{b,0}a^{-3} > \Omega_{de,0} \Rightarrow a < \sqrt[3]{\frac{\Omega_{b,0}}{\Omega_{de,0}}}$

From (30) and by plugging numerical values in, we have $t < t_3 = 9.4\times10^9$ years

For $0 < t < t_1$, radiation dominated the universe.

For $t_1 < t < t_3$, matter dominated the universe.

For $t > t_3$, dark energy dominated the universe.

Also, at $t = t_2$, the radiation's energy density decreased under the dark energy's density.

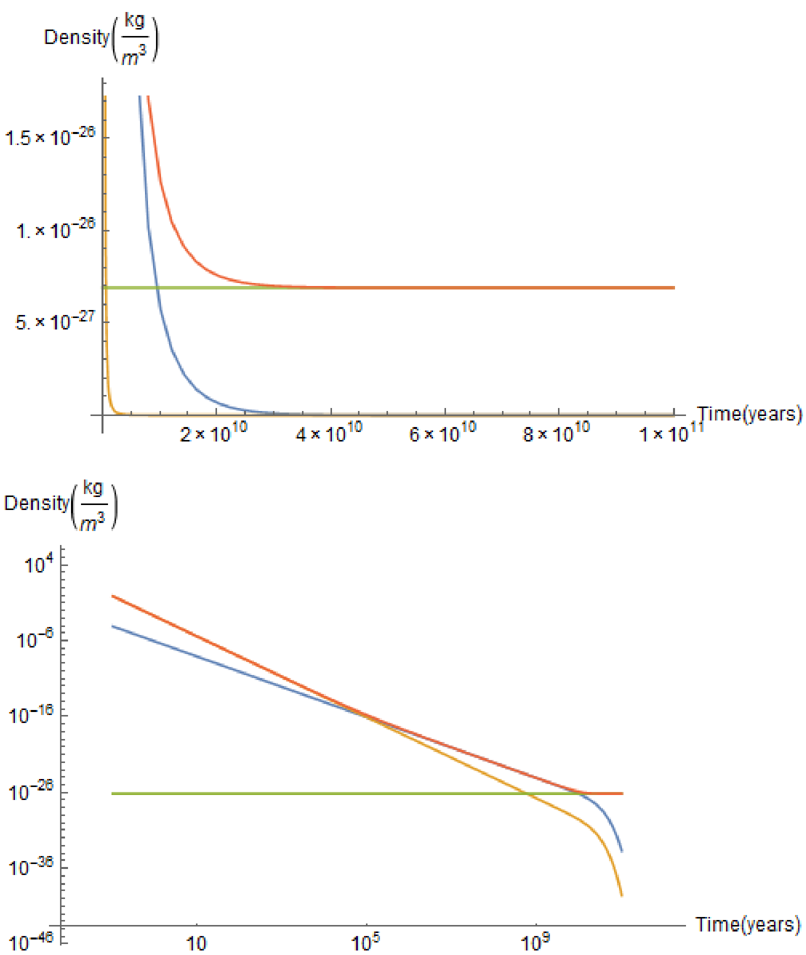

Figure 1. Plots of density against time (normal and logarithmic).

Radiation in orange, matter in blue, dark energy in greeen.

Total density in red.

3.4)

$\rho_{\text{dom}} = \rho_0 \Omega_{\text{dom},0} a^{-3(1+w_{\text{dom}})}$

$\rho_{\text{dom}} \gg \rho_0 \Omega_{de,0} a^0 \Rightarrow a^{-3(1+w_{\text{dom}})} \gg 1$

As we are interested in the far future, $t > t_0 \Rightarrow a(t) > 1$

Thus, we will have $-3(1+w_{\text{dom}}) > 0 \Rightarrow w_{\text{dom}} < -1$

This type of matter is called "phantom energy" . (citation wiki)

In this case, the universe will expand to an infinite size in a finite time[4] [5].

Expansion of our universe:

Now we shall try to derive a relation from which we can find H_0 :

Let's consider a photon traveling from the emission point to the observation point. The distance covered by the photon in the time dt is $dr = cdt$. We cannot simply sum those elementary distances because the space is expanding so we have to scale them. As such we have to sum $dr_0 = \frac{cdt}{a_{(t)}}$, where r_0 is the distance between the source of emission and the observer at the present time when $a = 1$.

$\int_0^{r_0} dr_0 = \int_{t_e}^{t_o} \frac{cdt}{a_{(t)}} \Rightarrow r_0 = \int_{t_e}^{t_o} \frac{cdt}{a_{(t)}} \Rightarrow r_{(t)} = a_{(t)} \int_{t_e}^{t_o} \frac{cdt}{a_{(t)}}$, where $r_{(t)}$ is the distance between the source of emission and the observer at a given time t.

$r_{(t)} = a_{(t)} \int_{t_e}^{t_o} \frac{cdt}{a_{(t)}} \frac{da}{da} \Rightarrow r_{(t)} = a_{(t)} c \int_{a_{(t_e)}}^{a_{(t_o)}} \frac{da}{a_{(t)} \frac{da}{dt}}$, where $a_{(t_e)} = \frac{1}{1+z_e}$ and $a_{(t_0)} = 1$.

From 1.3.b) we know that $a_{(t)} = \frac{1}{1+z} \Rightarrow da = -\frac{dz}{(1+z)^2} = -a_{(t)}^2 dz$

$\Rightarrow r_{(t)} = a_{(t)} c \int_{a_{(t_e)}}^{a_{(t_0)}} \frac{-a_{(t)}^2 dz}{a_{(t)} \frac{da}{dt}} = -a_{(t)} c \int_{a_{(t_e)}}^{a_{(t_0)}} \frac{dz}{\frac{\dot{a}_{(t)}}{a_{(t)}}} = -a_{(t)} c \int_{z_e}^{0} \frac{dz}{H_{(z)}} = a_{(t)} c \int_0^{z_e} \frac{dz}{H_{(z)}}$

We have to find $H_{(z)}$ now. Using the Friedmann equation $\left(H^2 - \frac{8\pi G\rho}{3}\right)a^2 = \frac{-\text{Kc}^2}{r_0^2}$ and expressions for the densities derived at 3.2) : $\left(H^2 - \frac{8\pi G\rho_0}{3}(\Omega_{b,0} a^{-3} + \Omega_{r,0} a^{-4} + \Omega_{\text{de},0})\right)a^2 = \frac{-\text{Kc}^2}{r_0^2}$. Because our universe is nearly flat[6] we can use K=0.

$\Rightarrow H^2 = \frac{8\pi G\rho_0}{3}(\Omega_{b,0} a^{-3} + \Omega_{r,0} a^{-4} + \Omega_{\text{de},0}) \Rightarrow H = H_0 \sqrt{\Omega_{b,0} a^{-3} + \Omega_{r,0} a^{-4} + \Omega_{\text{de},0}}$

$\Rightarrow H_{(z)} = H_0 \sqrt{\Omega_{b,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\text{de},0}}$

From the text of the problem we know that $D_L = \sqrt{\frac{L}{4\pi f}}$, but L is an instrinsic property of the source so it doesn't depend on the distance from it, but f does depend on the distance between the source and the observer which is r_0 as defined above.

$f = \frac{\Delta E_o}{\Delta t_o} \frac{1}{4\pi r_0^2}$, where $\frac{\Delta E_o}{\Delta t_o}$ is the energy in unit time as received by the observer. We can relate this quantity to L in the following way:

4. http://arxiv.org/pdf/hep-th/0610213v2.pdf

5. http://arxiv.org/pdf/astro-ph/0302506v1.pdf

6. http://map.gsfc.nasa.gov/universe/uni_shape.html

$\Delta E \sim \frac{1}{\lambda} \Rightarrow \Delta E_o = \Delta E_e \frac{\lambda_e}{\lambda_o}$. Using the formula for the cosmological redshift derived at 1.3.b) :

$\Rightarrow \Delta E_o = \frac{\Delta E_e}{1+z_e}$

Using the equation for cosmological time dilation also derived at 1.3.b) :

$\Delta t_o = (1+z_e)\Delta t_e$

Using the definition of luminosity as the energy radiated by the source in unit time at its surface: $L = \frac{\Delta E_e}{\Delta t_e}$ we now have $f = \frac{L}{4\pi[(1+z_e)r_0]^2}$. We also know that $D_L = \sqrt{\frac{L}{4\pi f}} \Rightarrow f = \frac{L}{4\pi D_L^2}$.

$\Rightarrow D_L = (1+z_e)r_0$

We now have 3 main equations from which we can find a formula for H_0:

$r_{(t)} = a_{(t)} c \int_0^{z_e} \frac{dz}{H_{(z)}} \Rightarrow r_{(t_o)} = r_0 = c \int_0^{z_e} \frac{dz}{H_{(z)}}$

$H_{(z)} = H_0 \sqrt{\Omega_{b,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\mathrm{de},0}}$

$D_L = (1+z_e)r_0$

$\Rightarrow r_0 = \frac{D_L}{1+z_e} = c \int_0^{z_e} \frac{dz}{H_0\sqrt{\Omega_{b,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\mathrm{de},0}}} \Rightarrow H_0 = \frac{c(1+z_e)}{D_L} \int_0^{z_e} \frac{dz}{\sqrt{\Omega_{b,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\mathrm{de},0}}}$

From this formula we can compute H_0 using the tables given.[7]

First method:

First we will linearize the data

Let $f(z_e) = (1+z_e) \int_0^{z_e} \frac{dz}{\sqrt{\Omega_{b,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\mathrm{de},0}}}$

Now, we can write $D_L = \frac{c}{H_0} f(z_e)$

7. general citation from int to modastro